Problem. Use Gauss-Jordan row reduction to solve the given system of equations.

$$\begin{aligned} x + y + 3z \qquad &= 12 \\ y + 4z + w &= 18 \\ x + 3y + 7z + 2w &= 30 \\ x + y + 3z + w &= 12 \end{aligned}$$

Solution. Remember that we can write this system of equations as a matrix multiplication problem by writing out the coefficients of each variable in the corresponding columns: x goes in column 1, y in column 2, z in column 3, and w in column 4.

$$\begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 4 & 1 \\ 1 & 3 & 7 & 2 \\ 1 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 12 \\ 18 \\ 30 \\ 12 \end{bmatrix}$$

We then write the augmented matrix by attaching the solutions column to our coefficient matrix and perform Gauss-Jordan elimination by doing row reductions.

$$\begin{bmatrix} 1 & 1 & 3 & 0 & 12 \\ 0 & 1 & 4 & 1 & 18 \\ 1 & 3 & 7 & 2 & 30 \\ 1 & 1 & 3 & 1 & 12 \end{bmatrix} \xrightarrow{R3=R3-R1} \begin{bmatrix} 1 & 1 & 3 & 0 & 12 \\ 0 & 1 & 4 & 1 & 18 \\ 0 & 2 & 4 & 2 & 18 \\ 1 & 1 & 3 & 1 & 12 \end{bmatrix} \xrightarrow{R4=R4-R1} \begin{bmatrix} 1 & 1 & 3 & 0 & 12 \\ 0 & 1 & 4 & 1 & 18 \\ 0 & 2 & 4 & 2 & 18 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R3=R3-2R2} \begin{bmatrix} 1 & 1 & 3 & 0 & 12 \\ 0 & 1 & 4 & 1 & 18 \\ 0 & 0 & -4 & 0 & -18 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R2=R2+R3} \begin{bmatrix} 1 & 1 & 3 & 0 & 12 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 & -18 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R2=R2-R4} \begin{bmatrix} 1 & 1 & 3 & 0 & 12 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -18 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R1=R1-R2} \begin{bmatrix} 1 & 0 & 3 & 0 & 12 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -18 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R3=\frac{1}{-4}R3} \begin{bmatrix} 1 & 0 & 3 & 0 & 12 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 9/2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R1=R1-3R3} \begin{bmatrix} 1 & 0 & 0 & 0 & -3/2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 9/2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Finally, this tells us our solution is

$$x = -3/2$$

$$y = 0$$

$$z = 9/2$$

$$w = 0$$

or

$$(x, y, z, w) = (-3/2, 0, 9/2, 0).$$

Problem. The matrix $\begin{bmatrix} 1 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$ is the result of performing row reduction on some augmented matrix A. Determine the solution to the system of equations.

Solution. Using the first problem as a guideline, this matrix represents the system

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

The row of zeros tells us the system is dependent. Specifically, this system gives the equation of a line: $x + 2y = 6$, which shows explicitly the dependence of y on x. Rearranging gives $y = -x/2 + 3$, so our solutions are of the form

$$(x, -x/2 + 3).$$

Problem. The matrix $\begin{bmatrix} 1 & 0 & 17 \\ 0 & 1 & 10 \end{bmatrix}$ is the result of performing row reduction on some augmented matrix A. Determine the solution to the system of equations.

Solution. Using the first problem as a guideline, this matrix represents the system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 17 \\ 10 \end{bmatrix}.$$

This system gives the equations $x = 17$ and $y = 10$, so our solution is simply

$$(17, 10).$$