Write the general solution of the given linear system in vector form or show that there is no solution.

$$\begin{cases} x_1 - 2x_2 + 3x_3 + 2x_4 = 3 \\ 3x_1 - 5x_2 + 7x_3 + 4x_4 = 6 \\ 2x_1 - 3x_2 + 4x_3 + 2x_4 = 3 \end{cases}$$

**Solution.** Let's write out the matrix equation:

$$\begin{bmatrix} 1 & -2 & 3 & 2 \\ 3 & -5 & 7 & 4 \\ 2 & -3 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}.$$

So we can form the augmented matrix

$$\begin{bmatrix} 1 & -2 & 3 & 2 & | \, 3 \\ 3 & -5 & 7 & 4 & | \, 6 \\ 2 & -3 & 4 & 2 & | \, 3 \end{bmatrix}$$

Let's first clear the first column of the second row using $R_2 \leftarrow R_2 - 3R_1$.

$$\begin{bmatrix} 1 & -2 & 3 & 2 & | \, 3 \\ 0 & 1 & -2 & -2 & | -3 \\ 2 & -3 & 4 & 2 & | \, 3 \end{bmatrix}$$

Conveniently, this places a 1 (a pivot) in the second row! Now we clear out the first column of row 3 with $R_3 \leftarrow R_3 - 2R_1$.

$$\begin{bmatrix} 1 & -2 & 3 & 2 & | \, 3 \\ 0 & 1 & -2 & -2 & | -3 \\ 0 & 1 & -2 & -2 & | -3 \end{bmatrix}$$

Now we see the second and third rows are actually linearly dependent! So we will have a general solution! We continue with $R_3 \leftarrow R_3 - R_2$.

$$\begin{bmatrix} 1 & -2 & 3 & 2 & | \, 3 \\ 0 & 1 & -2 & -2 & | -3 \\ 0 & 0 & 0 & 0 & | \, 0 \end{bmatrix}$$

Since we have leading 1s in rows 1 and 2, we are done! This tells us we will have free variables $x_3$ and $x_4$. The second row tells us

$$x_2 - 2x_3 - 2x_4 = -3, \text{ so}$$

$$x_2 = 2x_3 + 2x_4 - 3.$$

and the first row tells us

$$x_1 - 2x_2 + 3x_3 + 2x_4 = 3, \text{ so}$$

$$x_1 = 2x_2 - 3x_3 - 2x_4 + 3 = 2(2x_3 + 2x_4 - 3) - 3x_3 - 2x_4 + 3 = 4x_3 + 4x_4 - 6 - 3x_3 - 2x_4 + 3 = x_3 + 2x_4 - 3,$$

$$x_1 = x_3 + 2x_4 - 3.$$

So if we let $x_3 = a$ and $x_4 = b$, we get a general solution of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a + 2b - 3 \\ 2a + 2b - 3 \\ a \\ b \end{bmatrix}.$$

---