Problem. Let G be a triangle-free graph on $2n$ vertices and uv be an edge in G. Let H be the induced subgraph where we remove u and v from G. Explain why you remove at most $2n-1$ edges incident to the those vertices.

Solution. Recall that the *neighborhood* of v in G is the set of all vertices adjacent to v, i.e.

$$N_v(G) = \{v' \in G \mid v'v \text{ is an edge of } G.\}.$$

Let's rephrase our problem in terms of neighborhoods: If G is a triangle-free graph on $2n$ vertices and uv is an edge in G, then H is created by removing each edge in $N_u(G)$ and $N_v(G)$ from G, and then removing u and v. Showing that we remove at most $2n-1$ edges is equivalent to proving the equation

$$|N_u(G)| + |N_v(G)| \leq 2n - 1.$$

So we prove it by contradiction. Suppose that

$$|N_u(G)| + |N_v(G)| = 2n.$$

The most important thing to remember here is that uv is an edge in G! What that means is that if both u and v are connected to the same vertex, call it w, then we will have a triangle uvw. We can show that this equation forces us to have such a vertex in G, which implies that we have a triangle in G. This contradicts the fact that G is triangle-free, so we will be done! Now let's show why w has to exist.

If we wanted to avoid such a vertex w, then we'd want $N_v(G)$ and $N_u(G)$ to be disjoint. In fact, if $N_v(G) \cap N_u(G) \neq \emptyset$, then any vertex in the intersection will work as w. But we MUST have

$$N_v(G) \cap N_u(G) \neq \emptyset.$$

If every vertex in $N_v(G)$ and $N_u(G)$ were distinct, AND $|N_v(G)| + |N_u(G)| = 2n$, then that means we must have at least $2n+1$ vertices in G! But G only has $2n$ vertices by definition, so that's a contradiction!

Therefore, the equation $|N_v(G)| + |N_u(G)| = 2n$ cannot hold, so we must have

$$|N_v(G)| + |N_u(G)| < 2n,$$

i.e.

$$|N_u(G)| + |N_v(G)| \leq 2n - 1.$$