First Day Linear Algebra Motivations

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First and most important: Don't worry if you don't recognize some words here! Just focus on the ideas and we'll learn the specifics later.

1 Graphics and Image Processing

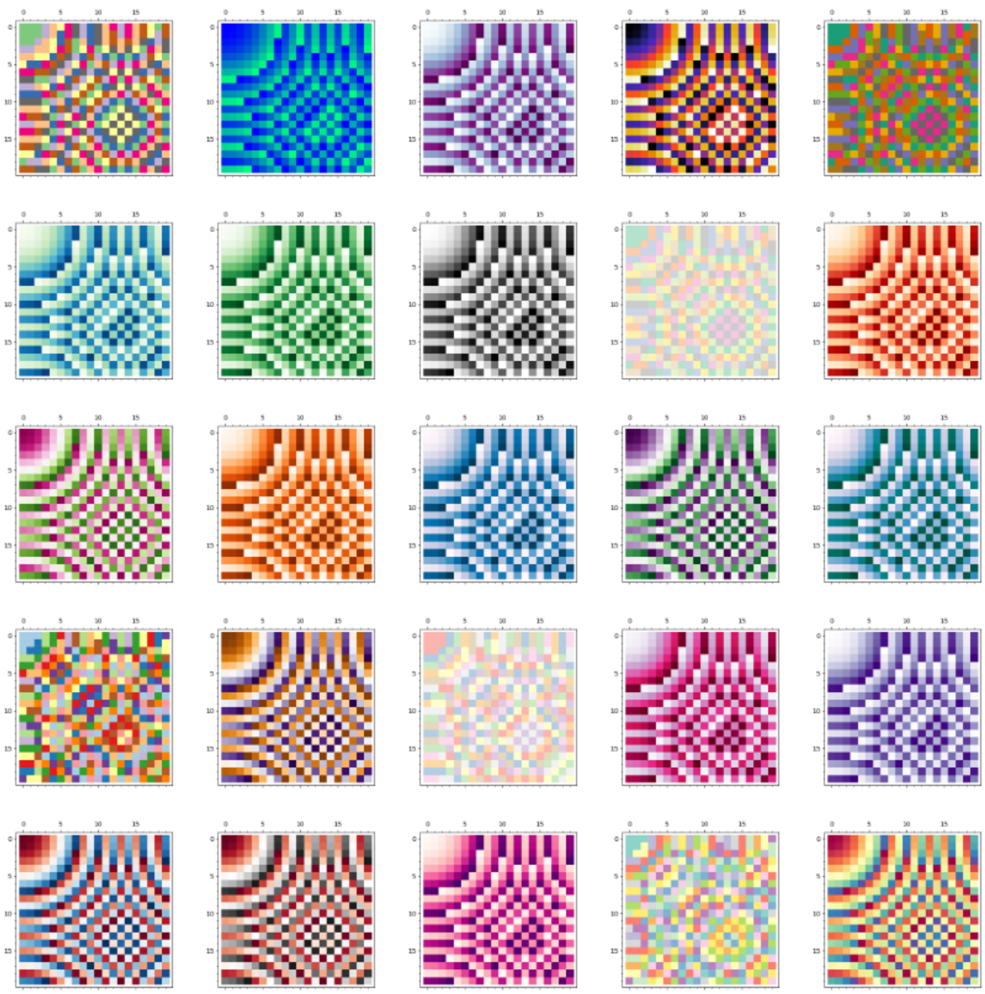
A major goal of Linear Algebra is to study ***linear transformations*** and one way we do this is with ***matrices (Singular - Matrix)***, which are 2-dimensional arrays of numbers. We can think of a picture as a 2-dimensional array of pixels, with each pixel containing some color. Putting these together, we can think of pictures as matrices! If we want to do some type of ***image processing***, then we can do this by manipulating the matrix.

1.1 Examples

First, here are a few examples from my blog at JMathG. Here is a large matrix:

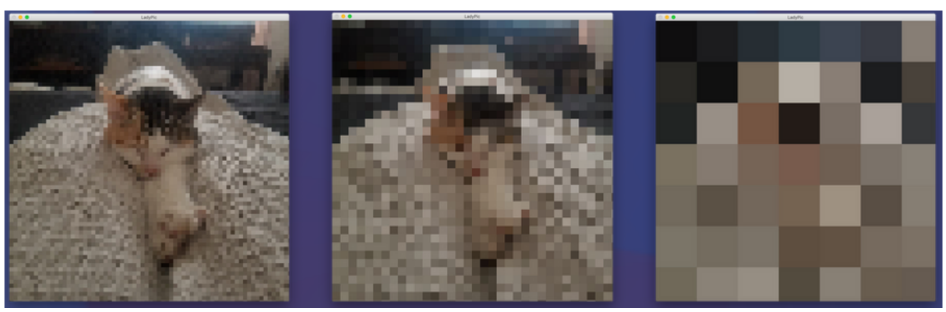
$$\begin{pmatrix}
0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 11 & 28 & 47 & 15 & 38 & 10 & 37 & 13 & 44 & 24 & 6 & 43 \\
1 & 2 & 5 & 10 & 17 & 26 & 37 & 50 & 12 & 29 & 48 & 16 & 39 & 11 & 38 & 14 & 45 & 25 & 7 & 44 \\
4 & 5 & 8 & 13 & 20 & 29 & 40 & 0 & 15 & 32 & 51 & 19 & 42 & 14 & 41 & 17 & 48 & 28 & 10 & 47 \\
9 & 10 & 13 & 18 & 25 & 34 & 45 & 5 & 20 & 37 & 3 & 24 & 47 & 19 & 46 & 22 & 0 & 33 & 15 & 52 \\
16 & 17 & 20 & 25 & 32 & 41 & 52 & 12 & 27 & 44 & 10 & 31 & 1 & 26 & 0 & 29 & 7 & 40 & 22 & 6 \\
25 & 26 & 29 & 34 & 41 & 50 & 8 & 21 & 36 & 0 & 19 & 40 & 10 & 35 & 9 & 38 & 16 & 49 & 31 & 15 \\
36 & 37 & 40 & 45 & 52 & 8 & 19 & 32 & 47 & 11 & 30 & 51 & 21 & 46 & 20 & 49 & 27 & 7 & 42 & 26 \\
49 & 50 & 0 & 5 & 12 & 21 & 32 & 45 & 7 & 24 & 43 & 11 & 34 & 6 & 33 & 9 & 40 & 20 & 2 & 39 \\
11 & 12 & 15 & 20 & 27 & 36 & 47 & 7 & 22 & 39 & 5 & 26 & 49 & 21 & 48 & 24 & 2 & 35 & 17 & 1 \\
28 & 29 & 32 & 37 & 44 & 0 & 11 & 24 & 39 & 3 & 22 & 43 & 13 & 38 & 12 & 41 & 19 & 52 & 34 & 18 \\
47 & 48 & 51 & 3 & 10 & 19 & 30 & 43 & 5 & 22 & 41 & 9 & 32 & 4 & 31 & 7 & 38 & 18 & 0 & 37 \\
15 & 16 & 19 & 24 & 31 & 40 & 51 & 11 & 26 & 43 & 9 & 30 & 0 & 25 & 52 & 28 & 6 & 39 & 21 & 5 \\
38 & 39 & 42 & 47 & 1 & 10 & 21 & 34 & 49 & 13 & 32 & 0 & 23 & 48 & 22 & 51 & 29 & 9 & 44 & 28 \\
10 & 11 & 14 & 19 & 26 & 35 & 46 & 6 & 21 & 38 & 4 & 25 & 48 & 20 & 47 & 23 & 1 & 34 & 16 & 0 \\
37 & 38 & 41 & 46 & 0 & 9 & 20 & 33 & 48 & 12 & 31 & 52 & 22 & 47 & 21 & 50 & 28 & 8 & 43 & 27 \\
13 & 14 & 17 & 22 & 29 & 38 & 49 & 9 & 24 & 41 & 7 & 28 & 51 & 23 & 50 & 26 & 4 & 37 & 19 & 3 \\
44 & 45 & 48 & 0 & 7 & 16 & 27 & 40 & 2 & 19 & 38 & 6 & 29 & 1 & 28 & 4 & 35 & 15 & 50 & 34 \\
24 & 25 & 28 & 33 & 40 & 49 & 7 & 20 & 35 & 52 & 18 & 39 & 9 & 34 & 8 & 37 & 15 & 48 & 30 & 14 \\
6 & 7 & 10 & 15 & 22 & 31 & 42 & 2 & 17 & 34 & 0 & 21 & 44 & 16 & 43 & 19 & 50 & 30 & 12 & 49 \\
43 & 44 & 47 & 52 & 6 & 15 & 26 & 39 & 1 & 18 & 37 & 5 & 28 & 0 & 27 & 3 & 34 & 14 & 49 & 33
\end{pmatrix}$$

And depending on how you decide to assign colors to each number, this matrix can represent many pictures!

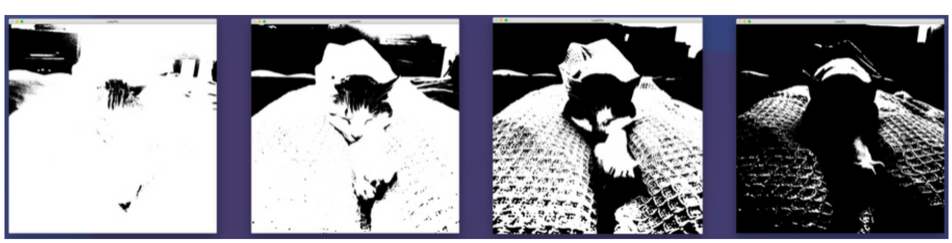


For a few more examples, we should start with this picture of my cat, which we load into Python (the programming language) as a matrix. If the picture is 1000×1000 pixels, then the matrix is a very large 1000×1000 matrix - but the computer can handle it! Then we can get different effects that we want on the picture by manipulating the matrix.

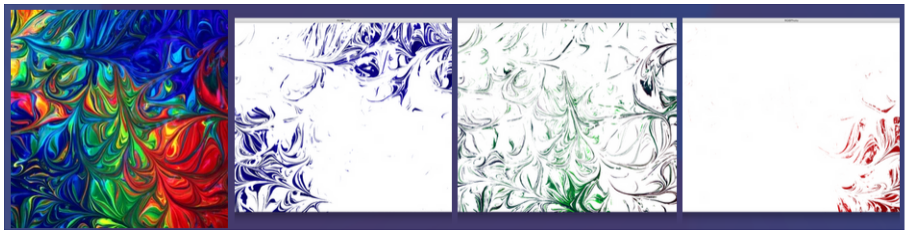
- We can pixelate the image to whatever precision we want:



- We can highlight spots of brightness/darkness we're interested in:



- And we can even isolate the Red-Blue-Green channels:



2 Animation/Computer Assisted Design (CAD)

The main character of Linear Algebra is the ***vector***, which can represent many different things. One you're most familiar with is that a vector has a starting point, a magnitude, and a direction. With this in mind, if we are playing a game or watching an animation, every object that we see can be thought of as a vector. But if our camera moves around, how does the game know the correct way to transform what we see to reflect that movement? To answer this, we use Linear Algebra!

We'll take our example from Vladimir Dobrushkin at Brown University. With every object being described by vectors (***Vector Graphics***), the way the objects are shifted, rotated, and scaled when the camera moves are all easily described via Linear Algebra, since these are all examples of linear transformations. Similarly, the way light interacts with the objects in our scenes can be described with Linear Algebra:

> "The first operation is transport, which means how much light is being shone on to the surface being rendered, from what direction this light is coming, and how intense the light source is. The second operation is called scattering which determines how the surface being rendered interacts with light."

3 Machine Learning/AI

Machine Learning and Artificial Intelligence seem very mystical at first, but after learning Linear Algebra, it becomes much clearer how it works. A great intro is at Visual Design. This is a *very* simplified explanation, but the computer "learning" is actually a process that involves solving large systems of equations, which is the first thing we'll talk about in this Linear Algebra course. Once we solve this system, we can check the output and compare it to our desired output, and then update the system so that our next attempt is hopefully closer to the desired output. Doing this repeatedly is how AI gets so good at whatever task it's doing.

For example, the famous chess AI named Deep Blue was developed in the 90's and eventually became good enough to beat the best Chess player in the world, Garry Kasparov.

4 Least-Squares Approximation

Remember when you had to find a "line of best-fit" going through some collection of data points? Well, take how interesting and useful that was and multiply it by 1000. Linear Algebra gives us a much much more general process where we can find a "best-fit" function for any set of data points (even if they're higher than 3-dimensional!). And the function can be anything you want.

Want to find the best cubic function going through some set of points? Go ahead! Want to find the best coefficients a, b, c so that the function

$$f(x) = a\cos(x) + b\sin(x) + ce^x$$

is as close as possible to describing some dataset? You got it!

5 Your Linear Algebra Applications Project

At the end of this class, you'll tell me some applications of Linear Algebra yourself! These projects include using Markov Chains to study cancer, creating amazing brackets for basketball, studying how Google's search algorithm works, and possibly more!

6 To The World of Pure Mathematics

This is a point of view that is definitely not standard to show first-time learners of Linear Algebra, but it's the area that I love the most, so I have to share it!

In the wide world of mathematics, there is a general split of "Applied Math" and "Pure Math" (although the divide becomes less and less defined as time goes on). Regardless of the many uses of Linear Algebra in Applied Math, it gives mathematicians in all different fields of Pure Math a common language to understand each other's work with.

6.1 Number Theory

Number Theory is an area of math where we study...numbers! We study prime numbers and expand the idea of what it even means to be prime. And we study numbers that form interesting sequences, which is what we'll do here now. One example of a really cool and unexpected connection between Linear Algebra and Number Theory arises with matrix multiplication giving us the Fibonnaci Numbers. There are more examples that we will see later in the course!

To start, the Fibonacci numbers are a sequence of numbers F_n that starts with $F_0 = 0$ and $F_1 = 1$, and then defines the next term in the sequence to be the sum of the two previous terms. So the Fibonacci numbers satisfy the *recurrence relation* $F_n = F_{n-1} + F_{n-2}$. Here's the start of this sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

The connection comes from starting with the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

and taking powers of it.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^4 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^5 = \begin{bmatrix} 8 & 5 \\ 5 & 3 \end{bmatrix}$$

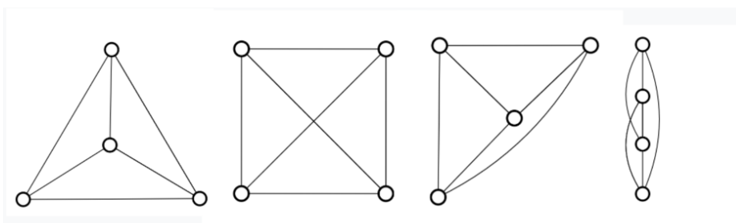
Looking at the entries of these matrices, we seem to see the Fibonacci numbers pop up quite a bit! Incredibly, the general rule is

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

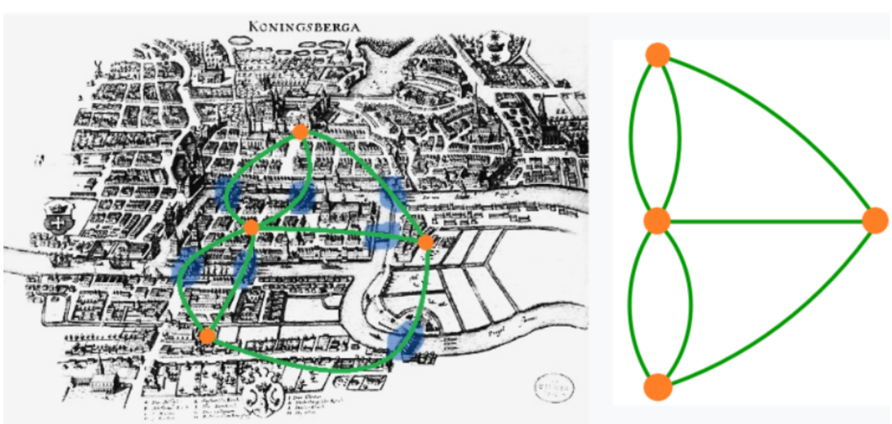
Once you feel comfortable with matrix multiplication, try to show ***why*** this is true!

6.2 Graph Theory

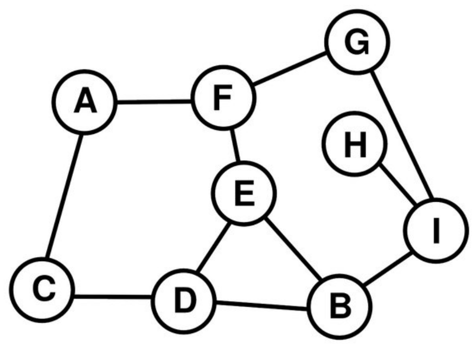
To a pure math crowd, when you say "graph", it's not the kind of graph that you've seen in Algebra or Calculus. Instead, it's what you might think of as a network: A bunch of points connected by edges. The points and edges can represent different things depending on what you're doing. For example, we can choose our points to be the people in this class, and different choices for edges will give different graphs. We could have an edge between two people if they are friends (Friendship Graph), or we could have an edge between two people if they live within 5 miles of each other. Or have an edge between two people if they went to the same middle school. Here are a few graphs:



The example that led to the creation of Graph Theory is ***The Bridges of Konigsberg***, which is described nicely at Xomnia. The city of Konigsberg was split by many rivers with 7 bridges connecting the portions together. A common riddle/question at the time was whether you could visit every portion of Konigsberg if you are only allowed to cross each bridge a single time. This long went unsolved, but Leonhard Euler famously turned the problem into a Graph Theory problem and proved definitively that you ***cannot*** do this. He presented this work on August 26, 1735, and this proof became known as the first theorem of graph theory. Here's a picture.



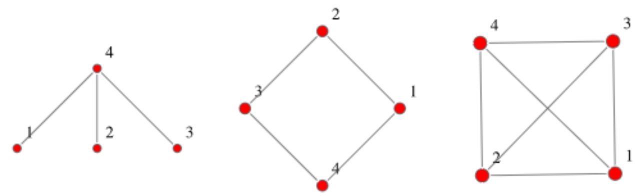
How does this relate to Linear Algebra? If we have a graph we're interested in, then we can use a matrix to encode that graph, and then linear algebra helps us discover things about our graph that would be really hard to find out otherwise. For example, if we label the vertices of this graph as follows:



Then we can draw a matrix whose rows and columns are indexed by the points A, B, C, D, E, F, G, H, and I. In the matrix, if we look at spot (i, j), then we'll write a 1 if points i and j are connected, and a 0 if not. This is a special matrix called a "(0,1)-matrix", because the entries are only 0 or 1. For example, in spot (A, F) and (A, C), we'll have a 1, while in spot (A, E), we'd have a 0. This is called the ***adjacency matrix*** of the graph. Here is the adjacency matrix of the above graph:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

And here are a few more simpler examples:



$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

There are many Linear Algebra techniques that we will learn that could help study these graphs. One that is easy to state right now is that taking powers of the adjacency matrix tells you the number of ways to move between two points only by going along the edges of our graph. Specifically, if A is our adjacency matrix, then the (i, j) entry of the matrix A^n will tell us the number of paths between points i and j of length n.

One thing this can be very helpful for is in determining efficient algorithms for deliveries (Travelling Salesman Problem). An important characteristic of a graph/network is if it has redundancies - multiple paths to get from one point to another. This is especially important if our graph represents connections (power lines) between transmission towers or internet connections between routers or WiFi-access-points, where losing connection could cause major issues (like at a school or hospital.

7 Game Theory

The classic example of this is the game SET, which this paper goes into so beautifully that I won't dwell on it here! The moral of the story is that this is a game that has been very popular for a long time and taking a Linear Algebra viewpoint reveals a lot of new information about this game that we didn't have before.

A good overview of how Linear Algebra helps us do Game Theory for other types of games is given by this student project. In particular, he discusses The Prisoner's Dilemma, which is a very popular example of this.