Graduate Research Plan

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We were playing a game. For a graph G, define a configuration of G to be an assignment of integers to every vertex, which we think of as placing some number of chips on each vertex. We *fire* a vertex v by sending a chip along each edge adjacent to v, and consider two configurations equivalent if one can be obtained by the other through chip-firing. The set of configurations that sum to 0 modulo this equivalence forms a finite abelian group $\mathcal{K}(G)$ called the *critical group* of G.[3]

During the summer of 2015, I did a research project describing the structure of the critical group of the square rook's graph.[4] Once the year started, I quickly became enthralled in a new application of a continuing research project relating three objects associated to a q-Weil polynomial $f \in \mathbb{Z}[T]$ (where q is a prime power) with a numerical equality:

(1) $|\mathcal{A}_3(\mathbb{F}_q, f)|$: The size of the isogeny class of principally polarizable ordinary abelian varieties of dimension 3 over $\mathbb{F}_q$ with characteristic polynomial of Frobenius f.
(2) h_K/h_{K^+}: The ratio of class numbers of the totally imaginary degree 6 number field $K = \mathbb{Q}[T]/(f)$ and its totally real cubic subfield K^+.
(3) $\nu_\infty(f)\prod_\ell \nu_\ell(f)$: The product over all rational primes ℓ of the relative frequency of f as the characteristic polynomial of an element of $\mathrm{GSp}_6(\mathbb{F}_\ell)$ and an Archimedean term coming from the Sato-Tate measure.

This research is inspired by work done on $\mathrm{GSp}_4(\mathbb{F}_\ell)$ and abelian varieties of dimension 2 by Achter and Williams[1] and work on $\mathrm{GL}_2(\mathbb{F}_\ell)$ and elliptic curves by Gekeler.[5]

1. My Research Plan

Could it be? Recently, I began to suspect a connection between my two research projects. Just as we define a divisor of an algebraic curve, we can define a divisor of a graph. We see that every configuration of G is in this sense a divisor of G, and the resulting Jacobian group (degree 0 divisors modulo linear equivalence) is the critical group.[2] This leads to the first part of my research plan: forming a connection between graphs and principally polarizable ordinary abelian varieties (PPOAV) through their Jacobian groups.

(R1): To a PPOAV A over $\mathbb{F}_q$, associate a graph G_A such that $\mathcal{K}(G_A) \simeq A(\mathbb{F}_q)$.

(R2): There already exists a notion of an isogeny graph for any PPOAV A. I want to study the critical group of this graph and see how it relates to A.

For number fields and the symplectic group, I know of no previously defined notion of a Jacobian. For number fields, we do have what is often called the Picard group: the ideal class group. We also have a Picard group for abelian varieties and graphs which is closely related to the Jacobian of those objects.

(R3): Construct a reasonable analogy to the Jacobian of abelian varieties for number fields and $\mathrm{GSp}_{2g}(\mathbb{F}_\ell)$.

Notice that in (2) we aren't dealing with h_K, which would be describing the Picard group of K, but the ratio h_K/h_{K^+}. This seems to be necessary whenever the dimension of our corresponding variety is greater than one. We can think of this ratio as describing the *relative class group* $\mathrm{Cl}(K/K^+)$, which is the kernel of the map $\mathrm{Cl}(K) \to \mathrm{Cl}(K^+)$ induced by the norm map.

(R4): Look for a relative structure in graphs, PPOAV, and the general symplectic group that might be analogous to the relative class group.

2. Intellectual Merit

I summarize my research plan in two main objectives.

(O1): Using the underlying Jacobian group structure on graphs, abelian varieties, number fields, and $\mathrm{GSp}_{2g}(\mathbb{F}_\ell)$, extend the numerical equalities between (1), (2), and (3) to higher dimensions.

(O2): After establishing the connection, translate combinatorial properties special to the critical group (critical configurations, minimal configurations, and poset structure, for example) to the other three objects.

This research would be of interest to mathematicians in multiple fields, and would provide new points of view on existing areas. Baker and Norine[2] proved a Riemann-Roch equivalent on graphs using this Jacobian interpretation. Musiker[6] formed a connection between elliptic curves and critical groups by creating an analogue of the Frobenius for graphs and defining a directed multigraph whose critical group was isomorphic to the group of $\mathbb{F}_{q^k}$-rational points on an elliptic curve, satisfying **(R1)** for dimension one.

Musiker poses a question at the end of this paper about creating an analogue for higher dimensional curves. This question is still open, and I believe that my background in the intersection of these two topics can help make progress in this direction, specifically with my idea **(R4)**.

3. Broader Impact

My long-term goal is to become a professor at a university where I can interact with and have an impact on the math community, both in a research setting and in a teaching setting. In terms of research, this plan has the potential to bring together mathematicians in multiple fields. Forming these connections between fields has been meaningful to me throughout my undergraduate career and will remain so moving forward.

In terms of teaching, I can see this plan being beneficial to undergraduates. As an undergraduate sophomore, I was able to do a research project on critical groups because of their accessibility. This means undergraduate research projects could be created and stated in terms of critical groups that could have important consequences for abelian varieties, number fields, and Lie groups like $\mathrm{GSp}_{2g}(\mathbb{F}_\ell)$. This would also serve as a good device to show students early on that the connections within mathematics are plentiful and beautiful.

I've been incredibly happy with the camaraderie and familial atmosphere within the math community at JMU, and wherever I end up at graduate school, I want to bring that feeling with me.

References

[1] J. Achter and C. Williams. Local heuristics and an exact formula for abelian surfaces over finite fields. *Canad. Math. Bull.*, 58(4):673–691, 2015.

[2] M. Baker and S. Norine. Riemann-Roch and Abel-Jacobi theory on a finite graph. *Adv. Math.*, 215(2):766–788, 2007.

[3] N. L. Biggs. Chip-firing and the critical group of a graph. *J. Algebraic Combin.*, 9(1):25–45, 1999.

[4] J. Ducey, J. Gerhard, and N. Watson. The smith and critical groups of the square rook's graph and its complement. *Electron. J. Combin.*, 23(4):P4.9, 19, 2016.

[5] Ernst-Ulrich Gekeler. Frobenius distributions of elliptic curves over finite prime fields. *Int. Math. Res. Not.*, (37):1999–2018, 2003.

[6] Gregg Musiker. The critical groups of a family of graphs and elliptic curves over finite fields. *J. Algebraic Combin.*, 30(2):255–276, 2009.