Personal Statement

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Going into my senior year of high school, I knew I wanted to study mathematics in college. I had an amazing teacher in high school who imprinted upon me the idea that math was about curiosity and exploration. I fell down the Wikipedia rabbit-hole and read posts I could not possibly understand on group theory, Galois theory, real analysis, and topology. Though I could not parse what exactly these fields were, I knew that they existed, and I knew I wanted to understand.

Now I am a senior at James Madison University, and I know I want to go to graduate school for mathematics. But the change between then and now is much more drastic than a simple continued desire to learn more math. In addition to my love of research, I have discovered a deep love of teaching. Through teaching, I find that I can share the beauty of mathematics with others and help dispel the stigma against math that so many people have.

1. Intellectual Merit

To me, the beauty of mathematics comes from its interconnections. Consider the following combinatorial problem:

> Take a list of numbers $1, 2, \ldots, n$. Choose two (say x, y) at random, remove them from the list, and add back in the number $x + y + xy$. Iterating this process ends with one number. What is that number?

We could proceed by induction on n, or we could take a route that is more fun. We can define a generating function $f(x) = (1 + x)(1 + 2x) \ldots (1 + nx)$, and notice that by Vieta's formulas with symmetric functions, $f(1) - 1 = (n + 1)! - 1$ is the number we end up with! Further, thinking of this as an Ehrhart polynomial, this process actually counts the number of lattice points in the n-dimensional cuboid with side lengths $1, 2, \ldots, n$. Just like that, we have a beautiful connection between combinatorics, algebra, and geometry.

This philosophy has guided me throughout my research experiences. My first opportunity to do research came freshmen year, when I approached a professor, Dr. Laura Taalman, asking for help with computing the Alexander-Conway polynomial of a figure-eight knot. After our discussion, she invited me to participate in a research project using 3D-printing to demonstrate knot invariants.

A few years later, in the spring of 2016, I participated in the Math in Moscow program. In addition to being the optimal opportunity to practice my Russian, this program also allowed me to take graduate courses I could not have taken at JMU. One of the most beautiful topics I encountered while there was Algebraic Topology. Remembering my first research project, I emailed Laura and suggested we try to visualize homotopies using 3D printing. This project will culminate in an invited session presentation at the Joint Mathematics Meetings 2017.

My second research experience was an REU at Michigan State University during the summer of 2014. This was my first exposure to combinatorics and I've adored the subject ever since. Under the guidance of Dr. Bruce Sagan, we brought together two previously-unlinked fields of combinatorics by studying the generating functions resulting from applying combinatorial statistics to avoidance classes of set partitions and restricted growth functions.

Back at school, I began doing research with Dr. Cassie Williams. At the start, I was simply computing the size of certain conjugacy classes of the general symplectic group $\mathrm{GSp}_6(\mathbb{F}_\ell)$. As I learned more about what we were doing, I became intrigued by the connections to number theory and geometry. Let A be a principally polarizable ordinary abelian variety of dimension g. The Frobenius endomorphism naturally lies in $\mathrm{GSp}_{2g}(\mathbb{Z}_\ell)$ and a theorem of Tate tells us that two such varieties are isogenous if and only if their characteristic polynomials are equal.

Therefore, we hope to count the size of an isogeny class of abelian varieties with characteristic polynomial $f(T)$. Doing so directly is infeasible, but making a (false) equidistribution assumption of Frobenius elements and computing the Euler product of the relative frequency of $f(T)$ as the characteristic polynomial of a matrix in $\mathrm{GSp}_{2g}(\mathbb{F}_\ell)$ over all rational primes ℓ (and an Archimedean term) ends up giving us exactly the size of the isogeny class, at least for $g = 1, 2$. Our project is essentially trying to establish this connection for $g = 3$.

In the summer of 2015, I did research with Dr. Josh Ducey at JMU on the *critical group* of a graph. Though we can formulate it as the cokernel of the Laplacian matrix L of the graph, I was drawn to the combinatorial interpretation: Define a configuration of your graph to be an assignment of integers to each vertex. We think of the integer assigned to each vertex as the number of chips that vertex holds, and we can *fire* a vertex v by sending one chip to each vertex adjacent to v.

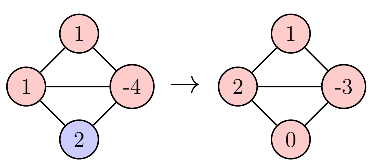

Figure 1. Two equivalent configurations by firing the blue vertex on the graph.

We consider two configurations equivalent if one can be obtained by the other through a series of chip-firing. The set of all configurations modulo this equivalence forms a group isomorphic to $\mathbb{Z} \times \mathcal{K}(G)$, where $\mathcal{K}(G)$ is a finite abelian group called the *critical group* of G.

I especially loved this project because of the relation between the algebraic view and the combinatorial view. If we want to find the critical group of a graph algebraically, then we need to put L in Smith normal form. But the way we attacked the problem was by working with actual configurations on our graph and finding a generating set for the whole group. For each equivalence class, there is a unique canonical representative, the critical configuration. Thinking in this manner allowed us to derive other structure from the critical group. For example, we have a natural poset structure where two critical configurations are related if one can add a single chip to one vertex and arrive at the other. We found that the height of this poset was actually $g + 1$ where $g = |E| - |V| + 1$ is the genus of the graph. We also counted the number of minimal configurations (i.e. minimal in the poset) of the n-wheel graph, $2^n - 2$.

However, thinking about the critical group in an algebraic way inspired my research plan. I feel I see a connection in the algebraic structures underlying the critical group, abelian varieties, the ideal class group of number fields, and possibly the general symplectic group. I hope to solidify this connection and use the nice combinatorial properties of the critical group to derive information in the other structures like the number of $\mathbb{F}_q$-rational points on the variety, the size of its isogeny class, the class number of the number field, and the size of conjugacy classes of the symplectic group.

2. Broader Impact

My research opportunities have also given me the opportunity to go to many math conferences and present on my work. I have taken every chance I could to talk with others interested in my work and learn about the work of others. I started by presenting on the knot theory and 3D printing project at my local MAA section meeting in 2014. Since then, I have presented at MathFest, the Joint Mathematics Meetings, Permutation Patterns, and the local undergraduate research conference called the Shenandoah Undergraduate Mathematics and Statistics Conference (SUMS), which brings together many schools in our area.

The math community has been so amazing to me, and because of this, I am constantly trying to give back. The math club at JMU, of which I am currently president, has flourished in the past few years. Our community has grown from a small group of ten or so people to often having 30-40 at each meeting. There are two major themes in our activities.

The first is a goal of building our community and helping others. We have Pi-day festivities every year, which generates a lot of interaction between the professors and students. In addition to being an enjoyable activity (which ends in the pieing of three professors and the math club executive board), we also donate a majority of the funds raised to good causes. This past year, we used those funds to support SUMS. The year before, we raised almost 500 dollars for the Charles C. Cunningham scholarship for outstanding tutors at JMU.

Another way that we have been able to build our community is by bringing underclassmen, who normally could not attend, to the JMM. We received a Student Government Association grant to bring two students last year and four students this year.

The other major theme is spreading our love of mathematics to others. One way we have done this is by visiting local middle schools and talking to them about fun topics in math. For example, on one occasion we talked about pattern finding in sequences, culminating in a pinecone painting activity to demonstrate the Fibonacci sequence. I also volunteered for the Expanding Your Horizons conference in March 2014 and 2015, which aims to encourage middle and high school aged girls to realize their potential and explore STEM fields.

My proudest achievement at JMU is being a co-creator and co-teacher of a class we call Math 167. The idea for this class spawned from a Math Club meeting back when the 10 or so of us could sit comfortably on the couches in the JMU math lounge. We wanted to show others the topics we thought were particularly beautiful and interesting, like combinatorics, abstract algebra, topology, graph theory, and game theory. These were topics we thought underclassmen math majors did not see soon enough and those who were not math majors might never see.

We spent the spring of 2015 drafting a syllabus, and with a massive amount of help from the math faculty, got our class on the books! We have one class per week for two hours, taught by two or three of the student teachers. We love to do interactive activities to demonstrate the topic of the week. For example, we played a group game of deal or no deal to demonstrate expected value and utility. For topology, we cut Mobius strips in various ways to see what the result was, and explored the Euler characteristic of various objects, eventually relating it to the genus of that object. The first semester we had over 30 students, many of whom have remained active in the community. A few have even become teachers of the class this semester! That's right, the class has run each semester since Fall 2015. I hope to return to JMU in five years, check the course catalog, and see that Math 167 is still being offered.

Wherever I end up, I want to continue making a difference in the math community and give others the opportunities that I have been so lucky to receive.