
Problem. Decide whether the subset A of $\mathbb{R}^3$ consisting of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with $|x+y+z| > 0$ is a subspace of $\mathbb{R}^3$.

Solution. There are three easy conditions that we can check to see if a subset W of a vector space V is a subspace:

(a) W is nonempty,

(b) W is closed under vector addition,

(c) W is closed under scalar multiplication.

Essentially, we just need W to also be a vector space. So if we can find anything that shows that it is not, then we're done.

In this example, the subset A is missing a very essential part of a vector space: the additive identity! The zero vector (0,0,0) does not satisfy $|x + y + z| > 0$, so is not a part of A. So A cannot be a vector space, and therefore is not a subspace.

Problem. Decide whether the subset S of $\mathbb{R}^4$ consisting of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ with $x_1 + x_2 + x_3 + x_4 = 0$ is a subspace of $\mathbb{R}^4$.

Solution. In this case, S is in fact a subspace! Let's prove it by showing that S satisfies those three conditions I listed above.

(a) S is nonempty, since it contains, for example, the zero vector. A non-trivial example of a vector in S is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$.

(b) Suppose (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) are vectors in S. Then $x_1 + x_2 + x_3 + x_4 = 0$ and $y_1 + y_2 + y_3 + y_4 = 0$. Their sum is $(x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4)$ and the sum of their components is

$$(x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) + (x_4 + y_4),$$

which we can rearrange as

$$(x_1 + x_2 + x_3 + x_4) + (y_1 + y_2 + y_3 + y_4)$$

and both terms in parentheses are equal to 0! So their sum is 0. Therefore, $(x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4)$ is in S, so S is closed under addition.

(c) If (x_1, x_2, x_3, x_4) is in S and $r \in \mathbb{R}$, then $r(x_1, x_2, x_3, x_4) = (rx_1, rx_2, rx_3, rx_4)$ and

$$rx_1 + rx_2 + rx_3 + rx_4 = r(x_1 + x_2 + x_3 + x_4) = r(0) = 0,$$

so (rx_1, rx_2, rx_3, rx_4) is in S, and S is closed under scalar multiplication.

All of these show that this subset S is in fact a subspace of $\mathbb{R}^4$.