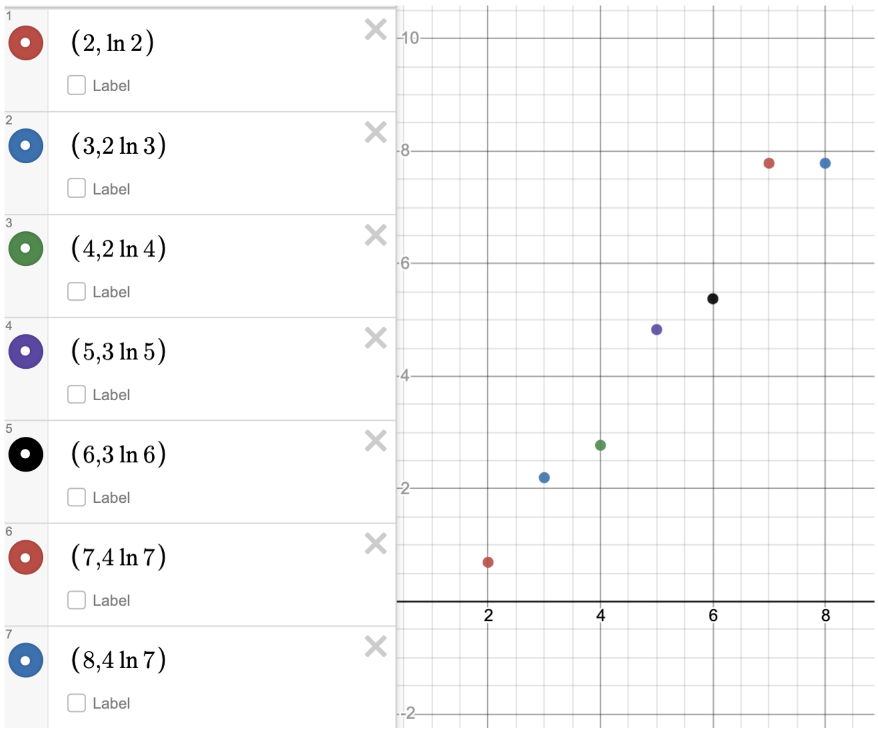
4. Consider the following points in $\mathbb{R}^2$.

$$(2, \ln(2)), \quad (3, 2\ln(3)), \quad (4, 2\ln(4)), \quad (5, 3\ln(5)),$$
$$(6, 3\ln(6)), \quad (7, 4\ln(7)), \quad (8, 4\ln(7)).$$

Find the equation for the least-squares line that approximates this data.

Solution. First, let's plot these points:



Notice that no straight line will pass through *every* one of these points. However, we can try to find coefficients A and B so that the line $y = Ax + B$ fits the points as closely as possible. We want to figure out A and B, so notice that plugging in $x = 2$ gives the equation $\ln(2) = 2A + B$. Plugging in $x = 3$ will give an equation $2\ln(3) = 3A + B$, and so on. Plugging in each $x = 2, 3, \ldots, 8$ gives six equations:

$$2A + B = \ln(2)$$

$$3A + B = 2\ln(3)$$
$$4A + B = 2\ln(4)$$
$$5A + B = 3\ln(5)$$
$$6A + B = 3\ln(6)$$
$$7A + B = 4\ln(7)$$

$$8A + B = 4\ln(7)$$

We can write this system of equations out in matrix form as

$$\begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \\ 6 & 1 \\ 7 & 1 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \ln(2) \\ 2\ln(3) \\ 2\ln(4) \\ 3\ln(5) \\ 3\ln(6) \\ 4\ln(7) \\ 4\ln(7) \end{bmatrix}$$

In cases like this, where we have too many equations (i.e. too many rows), the matrix equation $Mx = b$ isn't solvable. But to get the *least-squares solution*, we solve the equation $M^T M = M^T b$, where M^T is the transpose of M. With our matrices, this becomes

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \\ 6 & 1 \\ 7 & 1 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} \ln(2) \\ 2\ln(3) \\ 2\ln(4) \\ 3\ln(5) \\ 3\ln(6) \\ 4\ln(7) \\ 4\ln(7) \end{bmatrix}$$

Multiplying out, we get

$$\begin{bmatrix} 35 & 7 \\ 203 & 35 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \ln(2) + 2\ln(3) + 2\ln(4) + 3\ln(5) + 3\ln(6) + 4\ln(7) + 4\ln(7) \\ 2\ln(2) + 6\ln(3) + 8\ln(4) + 15\ln(5) + 18\ln(6) + 28\ln(7) + 32\ln(7) \end{bmatrix}$$

To simplify a little bit, we can apply the log rules $k\ln(a) = \ln(a^k)$ and $\ln(a) + \ln(b) = \ln(ab)$.

$$\begin{bmatrix} 35 & 7 \\ 203 & 35 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \ln(2 * 3^2 * 4^2 * 5^3 * 6^3 * 7^4 * 7^4) \\ \ln(2^2 * 3^6 * 4^8 * 5^{15} * 6^{18} * 7^{28} * 7^{32}) \end{bmatrix}$$

Or in simplest terms:

$$\begin{bmatrix} 35 & 7 \\ 203 & 35 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \ln(2^8 * 3^5 * 5^3 * 7^8) \\ \ln(2^{36} * 3^{24} * 5^{15} * 7^{60}) \end{bmatrix}$$

Remember that if we have an invertible 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its inverse is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. So we have

$$\frac{1}{-196}\begin{bmatrix} 35 & -7 \\ -203 & 35 \end{bmatrix} \begin{bmatrix} 35 & 7 \\ 203 & 35 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{-196}\begin{bmatrix} 35 & -7 \\ -203 & 35 \end{bmatrix} \begin{bmatrix} \ln(2^8 * 3^5 * 5^3 * 7^8) \\ \ln(2^{36} * 3^{24} * 5^{15} * 7^{60}) \end{bmatrix}$$

Which gives a final solution of

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \left(\frac{-35}{196}\right)\ln(2^8 * 3^5 * 5^3 * 7^8) + \left(\frac{7}{196}\right)\ln(2^{36} * 3^{24} * 5^{15} * 7^{60}) \\ \left(\frac{203}{196}\right)\ln(2^8 * 3^5 * 5^3 * 7^8) - \left(\frac{35}{196}\right)\ln(2^{36} * 3^{24} * 5^{15} * 7^{60}) \end{bmatrix}$$

which is approximately

$$\begin{bmatrix} A \\ B \end{bmatrix} \approx \begin{bmatrix} 1.25168 \\ -1.76785 \end{bmatrix}$$

To help visualize this, here is the line $y = 1.25168x - 1.76785$ along with the points, and you can see it's a very good fit!