
For convergence, we could be talking about either a sequence $\{a_n\}$ or its corresponding series $S = \sum_{n=1}^{\infty} a_n$. For sequences, this isn't too complicated:

- **A sequence $\{a_n\}$ converges if $\lim_{n\to\infty} a_n$ exists and is equal to a real number L.**

Another way to say this is:

- **A sequence $\{a_n\}$ diverges if $\lim_{n\to\infty} a_n$ does not exist or is equal to ∞.**

Examples

- The sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$, defined by $a_n = \frac{1}{n}$, is convergent.
- The sequence $1, 2, 3, 4, \ldots$, defined by $a_n = n$, diverges because $\lim_{n\to\infty} = \infty$.
- The sequence $0, 2, 0, 2, \ldots$, defined by $a_n = 1+(-1)^n$, diverges because $\lim_{n\to\infty} a_n = \text{DNE}$.

For series, the definition of convergence isn't complicated, but the methods of testing convergence are fairly vast.

- **A series $S = \sum_{n=1}^{\infty} a_n$ converges if the sequence of partial sums $S_k = \sum_{n=1}^{k} a_n$ converges**.

Examples

- The series $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, specifically to $\frac{\pi^2}{6}$ (The Basel Problem).
- The series $S = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, because it goes to ∞ (The Harmonic Series).
- The series $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, specifically to $\ln(2)$ (Alternating Harmonic Series).
- The series $S = \sum_{n=1}^{\infty} (-1)^n$ diverges, because the partial sums are $-1, 0, -1, \ldots$.

We have two types of convergence, conditional and absolute. These tell us about whether the series $S = \sum_{n=1}^{\infty} |a_n|$ converges.

- **If a series $S = \sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ does not converge, then we say that S is conditionally convergent.**
- **If a series $S = \sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that S is absolutely convergent.**

Examples

- If a series $S = \sum_{n=1}^{\infty} a_n$ has all positive terms, meaning $a_n > 0$ for all n, then $|a_n| = a_n$, so S is convergent if and only if it is absolutely convergent.
- The alternating harmonic series is the classic example of a conditionally convergent series, because $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges but $\sum_{n=0}^{\infty} \frac{1}{n+1}$ does not.
- The series $S = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$ converges, and so does $\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$, so S is absolutely convergent.

Now that we have the definitions down, let's talk about methods of testing convergence. Though the first method will actually help us determine if a series *diverges*. Throughout, we'll assume our series looks like $S = \sum_{n=1}^{\infty} a_n$

- **Divergence Test: If $\lim_{n\to\infty} a_n \neq 0$, or does not exist, then S diverges.**

This should make sense since if $\lim_{n\to\infty} a_n = L$, then the series S is eventually just going to look like $L + L + L + \dots$, which is ∞ unless $L = 0$.

Examples

- The series $S = \sum_{n=1}^{\infty} \frac{3n^2+1}{n^2-13n+7}$ diverges because $\lim_{n\to\infty} a_n = 3 \neq 0$.
- The series $S = \sum_{n=1}^{\infty} \sin(n)$ diverges because $\lim_{n\to\infty} a_n = \text{DNE}$.

It's important to note that the converse is NOT true. If $\lim_{n\to\infty} a_n = 0$, then the Divergence Test is inconclusive, and the series could converge or diverge.

Example

- The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

The next test tells us how to compare a series that we're interested in to one that we already know about. There are two ways to do so. Let $S = \sum_{n=1}^{\infty} a_n$ and $T = \sum_{n=1}^{\infty} b_n$

- **Direct Comparison Test: If $a_n \leq b_n$ for all n, then $S \leq T$, so**
  - **If T converges, then S converges.**
  - **If S diverges, then T diverges.**

There may be a case where $a_n \leq b_n$ does not hold for EVERY n, but the terms a_n seem to be generally smaller than b_n, as n gets bigger. This leads us to a less-strict comparison test.

- **Limit Comparison Test: If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, with c positive (i.e. nonzero and not ∞), then in the limit, $S \approx cT$, so either**
  - **S and T both converge, or**
  - **S and T both diverge.**

We have a particularly nice test when the series is an alternating series.
Let $S = \sum_{n=1}^{\infty} (-1)^n a_n$ be an alternating series (so $a_n \geq 0$ for all n).

- **Alternating Series Test: If $\lim_{n\to\infty} a_n = 0$, then S converges.**

Example

- This is a very quick way to show the alternating harmonic series converges. The sequence $\frac{1}{n}$ goes to 0 as $n \to \infty$, so the series $\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{n}\right)$ converges.

The next test is pretty clever! If we have a series $S = \sum_{n=1}^{\infty} a_n$, we can think of it as a Riemann sum approximation with $\Delta x = 1$ for some $f(x)$ with $f(n) = a_n$ for all n. If $f(x)$ is a decreasing function, then we know that the right-hand Riemann sum will be an underestimate. Therefore, if the integral $\int_1^{\infty} f(x)dx$ converges, the series S must also converge.

Similarly, the left-hand Riemann sum will be an overestimate, which means that if the integral $\int_1^{\infty} f(x)dx$ diverges, the series S must diverge too. Put together, we get

- **Integral Test: Let $f(x)$ be a decreasing function on $(1, \infty)$ that is non-negative and $f(n) = a_n$ for all n. Then $S = \sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.**

Example

- The integral test allows us to prove the convergence or divergence of a **p-series**, a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. The function $\frac{1}{x^p}$ is a decreasing function on $(1, \infty)$, so we can use the integral test. If $p = 1$, then

$$\int \frac{1}{x^p} = \ln(x)$$

so evaluating from $x = 1$ to ∞ diverges, which tells us the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

If $p > 1$, then

$$\int_1^{\infty} \frac{1}{x^p} = \frac{1}{1-p},$$

so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

The final two tests we'll talk about are based off the Geometric Series $\sum_{n=0}^{\infty} ar^n$ which converges only for $|r| < 1$. The idea is to take the series $S = \sum_{n=1}^{\infty} a_n$ and try to look at it as a geometric series of sorts.

- **Ratio Test: Let $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$. If $r < 1$, then S converges absolutely. If $r > 1$, then S diverges. If $r = 1$, the test is inconclusive.**
- **Root Test: Let $\lim_{n\to\infty} |a_n|^{1/n} = r$. If $r < 1$, then S converges absolutely. If $r > 1$, then S diverges. If $r = 1$, the test is inconclusive.**

Examples

- The Ratio test is very helpful when factorials are involved. Take $S = \sum_{n=1}^{\infty} \frac{e^n}{n!}$. Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{e^{n+1}/(n+1)!}{e^n/n!} = \lim_{n\to\infty} \frac{e}{(n+1)} = 0.$$

Since this is less than 1, we know S converges.

- The Root test is particularly helpful when our terms include a power of n. Take $S = \sum_{n=1}^{\infty} \frac{(2n)^n}{(3n)^{n+1}}$. First, we can manipulate this slightly to get

$$S = \sum_{n=1}^{\infty} \frac{(2n)^n}{(3n)^n}\left(\frac{1}{3n}\right) = \sum_{n=1}^{\infty} \left(\frac{2n}{3n}\right)^n \left(\frac{1}{3n}\right) = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \left(\frac{1}{3n}\right).$$

Now this almost looks like a geometric series. In fact, you could prove this converges by doing a Direct Comparison Test with a geometric series. But we'll do so using the Root Test.

$$\lim_{n\to\infty}\left(\left(\frac{2}{3}\right)^n\left(\frac{1}{3n}\right)\right)^{1/n}=\lim_{n\to\infty}\left(\frac{2}{3}\right)\left(\frac{1}{3n}\right)^{1/n}=\frac{2}{3}\lim_{n\to\infty}\left(\frac{1}{3n}\right)^{1/n}=\frac{2}{3}(1)=\frac{2}{3}.$$

Since this is less than 1, we know S converges.

Everything above was about proving whether a series converges or diverges. But if a series does converge, then we often also want to be able to describe what it converges to. So down below, I'll include a few examples of series that converge to certain functions. These are **Maclaurin Series** for the given function. So if we're given a series and can recognize the terms as corresponding to one of these series, we can find a closed-form function for that series and then evaluate the function at the corresponding x value.

- **Geometric Series:** $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $x \in (-1, 1)$.
- **Natural Log:** $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \ln(1+x)$ for $x \in (-1, 1]$.
- **Exponential:** $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ for all $x \in \mathbb{R}$.
- **Sine:** $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin(x)$ for all $x \in \mathbb{R}$.
- **Cosine:** $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos(x)$ for all $x \in \mathbb{R}$.
- **Natural Numbers:** $\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $x \in (-1, 1)$.
- **Squares:** $\sum_{n=0}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$ for $x \in (-1, 1)$.

Examples

Let $S = \sum_{n=0}^{\infty} \frac{3^n - 1}{6^n}$. Then

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{3^n - 1}{6^n} \\ &= \sum_{n=0}^{\infty} \frac{3^n}{6^n} - \frac{1}{6^n} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n \end{aligned}$$

Now we recognize that this is the form of two geometric series. The first has $x = \frac{1}{2}$ and the second has $x = \frac{1}{6}$. So

$$S = \frac{1}{1 - \frac{1}{2}} - \frac{1}{1 - \frac{1}{6}} = 2 - \frac{6}{5} = 0.8$$

Let's look at the alternating harmonic series $S = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$. We recognize that this is the form of $\ln(1+x)$, evaluated at $x = 1$ (notice $x = 1$ is included in the interval of convergence). Therefore,

$$S = \ln(2).$$

Let $S = \sum_{n=0}^{\infty} \frac{n^2 - n}{3^n}$. Then

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{n^2 - n}{3^n} \\ &= \sum_{n=0}^{\infty} \frac{n^2}{3^n} - \frac{n}{3^n} \\ &= \sum_{n=0}^{\infty} n^2 \left(\frac{1}{3}\right)^n - \sum_{n=0}^{\infty} n \left(\frac{1}{3}\right)^n \end{aligned}$$

So this is the difference of two series, with the first being the series for squares and the second being the series for the natural numbers. Both of them are evaluated at $x = \frac{1}{3}$. So

$$S = \frac{(1/3)(1 + 1/3)}{(1 - 1/3)^3} - \frac{1/3}{(1 - 1/3)^2} = \frac{3}{2} - \frac{3}{4} = \frac{3}{4}.$$

Let $S = \sum_{n=0}^{\infty} \frac{5}{2^n n!}$. Then

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{5}{2^n n!} \\ &= 5 \sum_{n=0}^{\infty} \frac{1}{2^n n!} \\ &= 5 \sum_{n=0}^{\infty} \frac{(1/2)^n}{n!} \end{aligned}$$

Now we recognize that this is the form of an exponential function, evaluated at $x = \frac{1}{2}$. So

$$S = 5e^{1/2}.$$