Problem 1. Prove that for any integer $m \geq 1$, $R(2,m) \leq \binom{m}{1}$.

Solution 1. Let's recall the definition of $R(r,s)$. Consider a complete graph on n vertices with its edges colored blue or red. Since there are two colors, there are $2^{\binom{n}{2}}$ possible colorings. Then $R(r,s)$ is the smallest value of n so that every one of these colorings contains a blue clique of size r or a red clique of size s. Remember a *clique* is a subgraph in which every vertex is connected, i.e. a subgraph that's also a complete graph.

So if we want to show that $R(2,m) \leq \binom{m}{1} = m$, we want to show that having $n = m$ vertices is *sufficient* to have the property of every 2-coloring containing a 2-clique colored blue, or an m-clique colored red. Then the minimum $n = R(2,m)$ must be smaller than m. So let's do that!

Remember that a 2-clique is simply two vertices connected by an edge! This means that if we try to avoid containing a blue 2-clique, we just can't color any edges blue! So we have to color each edge red. If we have $n = m$ vertices, then each edge is colored red, so we have a red m-clique. So if we have m vertices, any coloring will contain either a blue edge (a blue 2-clique) or a red m-clique. This shows that $R(2,m) \leq m$.

Problem 2. Prove that for any integer $m \geq 1$, $R(m,2) \leq \binom{m}{m-1}$.

Solution 2. Now this question can be seen in two ways. The first is quick using a few properties:

1. Ramsey numbers are symmetric: $R(r,s) = R(s,r)$.
2. $\binom{a}{b} = \binom{a}{a-b}$.

We apply both of these to problem 10 and we get it!

The other way is to build a graph like we did in problem 10. Now $R(m,2)$ is the smallest number of vertices n in which any edge-coloring of K_n with blue or red edges must contain a blue m-clique or a red 2-clique. Again, a red 2-clique is just a red edge! So to avoid this property, we must color every edge of K_n blue. But once $n = m$, this will be a blue m-clique, so K_m has the desired property! Therefore, $R(m,2) \leq m = \binom{m}{m-1}$.

Problem 3. Prove that for any integers $s, t > 0$, we have

$$R(s,t) \leq \binom{s+t+2}{s-1}.$$

Solution 3. To show this, we need to use this very helpful identity of Ramsey Numbers:

$$R(r,s) \leq R(s-1,t) + R(s,t-1).$$

To see why this holds, we do EXACTLY the same thing that we did in the previous two problems: work with a graph on $n = R(s-1,t) + R(s,t-1)$ vertices and show that it satisfies Ramsey's property for $R(r,s)$ (Explicitly, that every coloring of K_n contains a blue r-clique or a red s-clique).

We now go through a few cases to show that K_n does have this property. First, we choose any vertex v of K_n and look at the blue edges incident to it. Remember the total number of edges of edges incident to v is $n-1 = R(s-1,t)+R(s,t-1)-1$, so there are either

1. at least $R(s-1,t)$vertices connceted to v with edges colored blue, or

2. at least $R(s,t-1)$ vertices connected to v with edges colored red.

In the first case, we take the set $B = \{u \in K_n \mid \text{ the edge (v,u) is blue}\}$. Since the induced subgraph on the vertices in B has at least $R(s-1,t)$ vertices, we know it must have either an $(s-1)$-clique colored blue, or a t-clique colored red. In the first case, remember every edge (v,u) with $u \in B$ is colored blue, so adding v to the $(s-1)$-clique gives an s-clique colored blue! In the second case, having a red t-clique inside B means we have a red t-clique in all of K_n as well, since B was a subgraph!

This establishes that for colorings in case 1., the complete graph K_n on $n = R(s-1,r)+R(s,t-1)$ vertices will always contain either a blue s-clique or a red t-clique, and therefore,

$$R(s,t) \leq R(s-1,t)+R(s,t-1).$$

And in fact, addressing case 2. amounts to essentially swapping the words red and blue! This again goes back to the fact that Ramsey Numbers are symmetric. So overall, this establishes the inequality.

OK, so now that we have that, we can begin our proof by induction. But we don't induct on s and t separately, we take a clever step and induct on $s+t$. The reason for this comes from the binomial coefficient identity:

$$\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1}.$$

My favorite way to remember this identity is the following. Suppose we have a items and we want to choose b of them. Choose a special object X. Then our subset of size b either doesn't contain X, in which case we still need b items, but only have $a-1$ to choose from (removing X). This accounts for the $\binom{a-1}{b}$. The other option is that the subset does contain X. Then we only need $b-1$ more objects from the $a-1$ remaining, which accounts for the $\binom{a-1}{b-1}$.

Now the induction is actually pretty quick! We can check the base case $s+t=2$ directly. Recall a 1-clique is simply a vertex, so every graph except the empty graph satisfies this Ramsey property. So having $s=t=1$ to get $s+t=2$ means we will have $R(s,t)=1$. And $\binom{s+t-1}{s-1} = \binom{0}{0} = 1$, so the inequality holds.

The induction hypothesis tells us to assume the main statement holds for $s+t=k$, and the inductive step is to prove that it holds true for $s+t=k+1$. We first apply the Ramsey inequality we proved a moment ago:

$$R(s,t) \leq R(s-1,t)+R(s,t-1).$$

Now if $s+t=k+1$, then $(s-1)+t=k$, so the inductive hypothesis holds for $R(s-1,t)$. Therefore,

$$R(s,t) \leq \binom{s+t-3}{s-2} + R(s,t-1).$$

Similarly, $s+(t-1)=k$, so the inductive hypothesis holds for $R(s,t-1)$, meaning

$$R(s,t) \leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1}$$

The final trick is to notice that this EXACTLY matches the binomial coefficient identity we showed earlier, just set $a=s+t-2$ and $b=s-1$. So

$$\binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}.$$

Combining this with the previous line gives the desired main theorem!

$$R(s,t) \leq \binom{s+t-2}{s-1}.$$