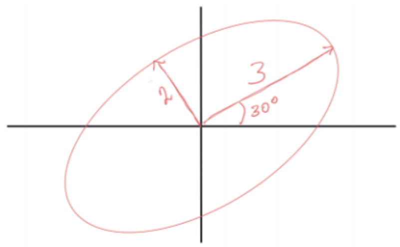
1. **2 × 2 matrix makes a circle to an ellipse, and we could use SVD to find the ellipse. If we do the reverse now. Consider the ellipse illustrated below, which has semimajor axis of length 3 and at a 30-degree angle with the x-axis, and has semi-minor axis of length 2.**




   a) Find the matrix A, which represents the transformation that will take the unit circle to this ellipse. (Maybe like transformation as a composition of rotations and scaling)

   b) Compute the area by the ellipse by the determinant

   c) R is a region of the plane with area equal to 7 units$^2$, Find the area of the image of this region after it undergoes the transformation represented by matrix 2A?

**Solution 1a.** Let's take the hint and write $A$ as a composition of a scalar matrix and a rotation matrix. A matrix that scales the x-axis $(1, 0)$ by $a$ and scales the y-axis $(0, 1)$ by $b$ is given by

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

A matrix that rotates the plane by $\theta$ is given by

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Composition of functions is multiplication of matrices, so we get

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}.$$

$$A = \begin{bmatrix} \frac{3\sqrt{3}}{2} & -\frac{3}{2} \\ 1 & \sqrt{3} \end{bmatrix}.$$

**Solution 1b.** Remember that a linear transformation scales area uniformly: If $T$ is a linear transformation and $R$ is a finite region in the plane, then

$$area(T(R)) = det(T)area(R).$$

Therefore, the area of the ellipse the determinant of $A$ multiplied by the area of the unit circle, which is $\pi$. This is

$$\pi \det A = \pi \det \begin{bmatrix} \frac{3\sqrt{3}}{2} & -\frac{3}{2} \\ 1 & \sqrt{3} \end{bmatrix} = \pi(9/2 + 3/2) = 6\pi.$$

**Solution 1c.** We just talked about that really helpful theorem in the last problem! We know

$$area(2A(R)) = det(2A)area(R) = 2^2 \det(A)area(R) = (4)(6)(7) = 168.$$

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2. Vectors u and v are a basis $\mathcal{B}$ for $\mathbb{R}^2$
   T is a transformation R2 →R2 that maps u → 3u, and v → u+3v.
   Consider the matrix A such that T(x) = Ax.

a) Consider this geometrically, identify one eigenvalue λ for matrix A, and describe the eigenspace in terms of u and v, what is the geometric multiplicity of λ?

b) Find the matrix B, which the $\mathcal{B}$ matrix for $T$.

c) What is the relationship between the eigenvalues of A and the eigenvalues of B? (Use the answer from part b, explain why the matrix A only has 1 eigenvalue)

d) What can you say about algebraic multiplicity of the eigenvalue λ from part (a)? Is matrix A diagonalizable?

**Solution 2a.** Geometrically, the description makes it clear that $u$ is an eigenvector with eigenvalue 3. To get a clearer picture, we can take a linear combination

$$au + bv \mapsto a(3u) + b(u + 3v) = (3a + b)u + (3b)v.$$

Then assuming $au + bv \mapsto \lambda(au + bv)$ tells us $\lambda a = 3a + b$ and $3b = \lambda b$. The second equation tells us that if $b \neq 0$, then $\lambda = 3$. Then the first equation tells us that $3a = 3a + b$ so $b = 0$, which contradicts what we just supposed. In other words, we have the obvious eigenvector $u$, and then no other (independent) linear combination gives an eigenvector. So the geometric multiplicity of $\lambda = 3$ is just 1.

**Solution 2b.** The fact that $u \mapsto 3u$ tells us that the first column of $B$ is $(3, 0)$ and the fact that $v \mapsto u + 3v$ tells us that the second column of $B$ is $(1, 3)$. Therefore,

$$B = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}.$$

**Solution 2c.** A very important property of eigenvalues is that they're independent of the basis you choose, so the eigenvalues of $A$ are the eigenvalues of $B$. This explains why $A$ only has one eigenvalue. The characteristic polynomial of $B$ is

$$f_B(x) = \det(xI - B) = \det \begin{bmatrix} x-3 & -1 \\ 0 & x-3 \end{bmatrix} = (x-3)^2,$$

so $B$ has one eigenvalue $\lambda = 3$. Therefore $A$ has one eigenvalue $\lambda = 3$.

**Solution 2d.** Although the set of eigenvalues of $A$ and $B$ might be the same, the multiplicities can differ. This is the case here, where we have a geometric multiplicity of 1, but $f_B(x) = (x-3)^2$, so the algebraic multiplicity of $\lambda$ is 2, the exponent.

**3. Consider the following matrix**

$$A = \begin{bmatrix} 11 & -5 \\ -2 & 10 \end{bmatrix}$$

**a) Compute singular values for matrix A.**

**b) Compute SVD for matrix A**

**c) Given equation $A = U\Sigma V^T$, prove that, if A is invertible, then $A^{-1} = V\,\Sigma^{-1}U^T$.**

**d) Use the combination of your work above to find the singular values for $A^{-1}$?**

**Solution 3a.** Recall the singular values of a matrix $A$ are the square-roots of the eigenvalues of the matrix $A^TA$. We have

$$A^TA = \begin{bmatrix} 11 & -2 \\ -5 & 10 \end{bmatrix} \begin{bmatrix} 11 & -5 \\ -2 & 10 \end{bmatrix} = \begin{bmatrix} 11^2 + 2^2 & -55-20 \\ -55-20 & (-5)^2 + 10^2 \end{bmatrix} = \begin{bmatrix} 125 & -75 \\ -75 & 125 \end{bmatrix}.$$

Its characteristic polynomial is

$$f(x) = \det(xI - A^TA) = \det \begin{bmatrix} x-125 & 75 \\ 75 & x-125 \end{bmatrix} = x^2 - 250x + 10000 = (x-200)(x-50),$$

so we see the eigenvalues of $A^TA$ are $\lambda_1 = 200, \lambda_2 = 50$. Which means the singular values of $A$ are

$$\sigma_1 = \sqrt{200} = 10\sqrt{2}, \qquad \sigma_2 = \sqrt{50} = 5\sqrt{2}.$$

**Solution 3b.** We want to find orthogonal matrices $U$ and $V$ so that $A = U\Sigma V^T$, where

$$\Sigma = \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}$$

is the *singular matrix*. Remember that the matrix $V$ is the matrix of eigenvectors, so let's calculate those: For $\lambda_1 = 200$, $A \begin{bmatrix} x \\ y \end{bmatrix} = 200 \begin{bmatrix} x \\ y \end{bmatrix}$ becomes

$$\begin{bmatrix} 125x - 75y \\ -75x + 125y \end{bmatrix} = \begin{bmatrix} 200x \\ 200y \end{bmatrix}.$$

The first and second rows tells us that $-75x - 75y = 0$, so $y = -x$, so our eigenvector is

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

For $\lambda_2 = 50$, $A \begin{bmatrix} x \\ y \end{bmatrix} = 50 \begin{bmatrix} x \\ y \end{bmatrix}$ becomes

$$\begin{bmatrix} 125x - 75y \\ -75x + 125y \end{bmatrix} = \begin{bmatrix} 50x \\ 50y \end{bmatrix}.$$

From this, we get $75x - 75y = 0$, so $x = y$. This means our eigenvector is

$$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

If we normalize these, we get the columns of $V$! So

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

To find $U$, we can solve for it in the equation $A = U\Sigma V^T$, so

$$U = AV\Sigma^{-1} = \begin{bmatrix} 11 & -5 \\ -2 & 10 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{10\sqrt{2}} & 0 \\ 0 & \frac{1}{5\sqrt{2}} \end{bmatrix}.$$

$$U = \begin{bmatrix} 11 & -5 \\ -2 & 10 \end{bmatrix} \begin{bmatrix} \frac{1}{20} & \frac{1}{10} \\ -\frac{1}{20} & \frac{1}{10} \end{bmatrix}$$

$$U = \begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix}.$$

This gives us our final answer: The SVD of $A$ is

$$\begin{bmatrix} 11 & -5 \\ -2 & 10 \end{bmatrix} = \begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix} \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

**Solution 3c.** Remember that $A^{-1}$ is by definition the matrix so that $AA^{-1} = I$. So let's check that $V\Sigma^{-1}U^T$ satisfies this equation:

$$A(V\Sigma^{-1}U^T) = (U\Sigma V^T)(V\Sigma^{-1}U^T) = U\Sigma(V^T V)\Sigma^{-1}U^T = U\Sigma^{-1}\Sigma U^T = UU^T = I.$$

Remember that $VV^T = UU^T = I$ because $U$ and $V$ are orthogonal matrices. This shows that $A^{-1} = V\Sigma^{-1}U^T$.

**Solution 3d.** Notice that if $A^{-1} = V\Sigma^{-1}U^T$, then $A^{-1}$ is in SVD since $V$ and $U^T$ are orthogonal matrices and $\Sigma^{-1}$ is a diagonal matrix. This means the singular values of $A^{-1}$ are just the inverses of the singular values of $A$:

$$\delta_1 = \frac{1}{\sigma_1} = \frac{1}{10\sqrt{2}}, \qquad \delta_2 = \frac{1}{\sigma_2} = \frac{1}{5\sqrt{2}}.$$