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**Problem.** Determine the matrix $P$ that diagonalizes the matrix $A$ and write the diagonal matrix.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

**Solution.** Let's review some of what we know about diagonalizing matrices. The matrix $A$ has three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with corresponding eigenvectors $v_1, v_2, v_3$. Remember that in equation-form, this means

$$Av_1 = \lambda_1 v_1$$

$$Av_2 = \lambda_2 v_2$$

$$Av_3 = \lambda_3 v_3$$

Now the point of linear algebra is to take a system of equations and turn it into a matrix equation. In this case, we can define a matrix $P$ whose columns are the eigenvectors $v_1, v_2, v_3$

$$P = \begin{bmatrix} \\ v_1 & v_2 & v_3 \\ \\ \end{bmatrix}$$

and then we know that multiplying by $A$ just scales each column by the associated eigenvector, which gives the matrix equation

$$AP = P \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Moving the P to the other side gives the equation

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

And now we've gotten back to the first definition of diagonalization that we have: **The matrix $A$ is diagonalizable if there exists a matrix $P$ so that $P^{-1}AP$ is a diagonal matrix**. But we now know that that diagonal matrix is exactly the matrix of eigenvalues and the transformation matrix $P$ is just the matrix of eigenvectors!

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Now that we know that, we just compute the eigenvalues and eigenvectors of $A$. We can compute the eigenvalues by calculating the characteristic polynomial:

$$\det(xI - A) = \det \begin{bmatrix} x-2 & -2 & -1 \\ -1 & x-3 & -1 \\ -1 & -2 & x-2 \end{bmatrix} = x^3 - 7x^2 + 11x - 5 = (x-1)^2(x-5).$$

So we see that $A$ has eigenvalues $5, 1, 1$. For $\lambda_1 = 5$, we can let $v_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and solve the equation $Av_1 = 5v_1$, which is

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 5 \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} 2a + 2b + c \\ a + 3b + c \\ a + 2b + 2c \end{bmatrix} = \begin{bmatrix} 5a \\ 5b \\ 5c \end{bmatrix}$$

To finish this, we can do a few different things. For example, subtracting the second row from the first gives $a - b = 5(a - b)$. If $a \neq b$, then this implies $5 = 1$, which is definitely not true! So we must have $a = b$. Subtracting the third row from the second gives $b - c = 5(b - c)$, so for the same reason, we require $b = c$. Therefore, for $\lambda_1 = 5$, we get eigenvector $\begin{bmatrix} a \\ a \\ a \end{bmatrix}$ for some non-zero $a$. Since we just need one eigenvector, let's take the easiest $a = 1$ which gives

$$\lambda_1 = 5, \qquad v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For the eigenvalue 1, we set up the same equation, but with a 1 instead of a 5.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} 2a + 2b + c \\ a + 3b + c \\ a + 2b + 2c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

This time, subtracting the second row from the first doesn't give any information - it says that $a - b = a - b$, which is true for any $a$ and $b$. This means $a$ and $b$ are *free variables*. We will have two free variables for $\lambda = 1$, since its multiplicity as an eigenvalue is 2, whereas the multiply of $\lambda = 5$ was 1, which is why we just end up with the free variable $a$.

Since $a$ and $b$ are free, the last equation $a + 2b + 2c = c$ can be rearranged to tell us $c = -a - 2b$, so our eigenvectors look like

$$\begin{bmatrix} a \\ b \\ -a - 2b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$

So we can take

$$\lambda_2 = \lambda_3 = 1, \qquad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$

Now we have our three eigenvalues

$$\lambda_1 = 5, \lambda_2 = 1, \lambda_3 = 1$$

with our three eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix},$$

so we know that

$$P^{-1}AP = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

where

$$P = \begin{bmatrix} \\ v_1 & v_2 & v_3 \\ \\ \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -2 \end{bmatrix}.$$

I highly encourage you to actually work out the multiplication $P^{-1}AP$ to see that it's not just a bunch of theory but a tangible calculation to get a diagonal matrix!